STOCHASTIC MODELS FOR RELIABILITY, AVAILABILITY, AND MAINTAINABILITY

Daniel F. Silva, Ph.D.
Assistant Professor
Industrial and Systems Engineering
Auburn University

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Outline

- Introduction to Markov models
- Examples of Markov chains in RAM
- Extensions and more complex models
- Computational tools for Markov models
- International standards for Markov models in RAM
- Summary & Conclusion

Daniel F. Silva,
silva@auburn.edu

INTRODUCTION TO MARKOV MODELS

What Is a Stochastic Process?

- A mathematical representation of a system that evolves **over time**, subject to **random variation**.

- The **state** of the system at a given time is captured by a random variable X_n or $X(t)$

- Stochastic processes can evolve in:
  - Discrete-time $\{X_n, \;\; n = 0,1,2 \ldots\}$
  - Continuous-time $\{X(t), \;\; t \geq 0\}$

Daniel F. Silva,
silva@auburn.edu

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Examples of Stochastic Processes

- Stock price at closing of trading day
- Stock price at any given time
- Inventory of multiple items at the end of each day
- Traffic in a website at any given time
- Failure state of a component at any given time
- Failure state of several component at any given time

Daniel F. Silva,
silva@auburn.edu

Applications of Stochastic Processes

- Revenue management
- Inventory planning
- Google's search engine
- Call center staffing
- Derivatives pricing
- Reliability, availability and maintainability

Daniel F. Silva,
silva@auburn.edu

Defining a Stochastic Process

- States
  - What are the states the system can occupy?
- Events
  - What can happen in the system that triggers a change in the state?
- Probabilities
  - What are the odds of events happening?

Daniel F. Silva,
silva@auburn.edu

Markov Chains

- Markov chains are a class of stochastic process
- MCs have a discrete (countable) state space S
  - **Finite**
  - Or infinite
  - E.g. $\{0,1\}, \{a, b, c\}, \{0,1,2,3, \ldots \infty\}$
- MCs can evolve in
  - Discrete time
  - **Continuous time**

Daniel F. Silva,
silva@auburn.edu

Markov Chains

- MCs have the "Markov Property":

$$P\{X(t_0 + s)|X(t), \forall t \leq t_0\} = P\{X(t_0 + s)|X(t_0)\}$$

- In words, the **future** only depends on the **present** and not on the **past**.

- In continuous time MCs, times between events follow an **exponential distribution**.

Daniel F. Silva,
silva@auburn.edu

The Exponential Distribution

- If random variable X is **exponential**, then:
  - $P\{X > t\} = e^{-\lambda t}$
  - $E[X] = \frac{1}{\lambda}$
  - $P\{X > t + s | X > s\} = P\{X > t\} = e^{\lambda t}$
  - This is called the **memorylessness property.**

Daniel F. Silva,
silva@auburn.edu

Markov Chains Dynamics

- The MC starts out at each state $i \in S$ with probability $a(i)$ at time $t = 0$.
- The MC **remains** at state i for an exponentially distributed amount of time, with parameter λ_i.
- After that the MC **transitions** to a new state j, with each state $j \in S$ having a transition probability p_{ij}.
- And so on…

Daniel F. Silva,
silva@auburn.edu

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Defining a Markov Chain

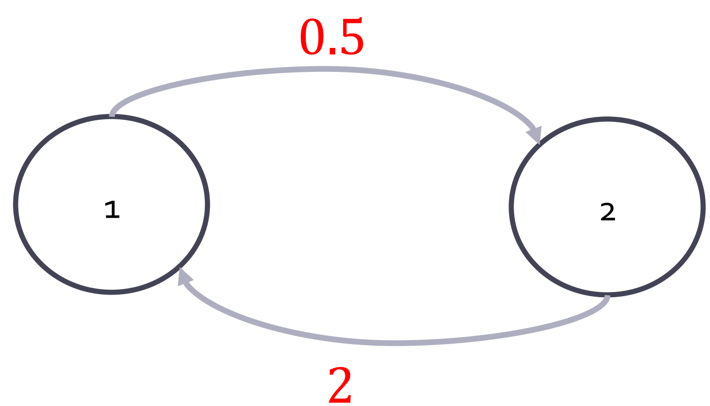
- State space: S
  - All possible states the system can occupy
- Initial distribution vector: a
  - $a(s)$ =Probability that the system is at state s at $t = 0$
- Generator matrix: Q
  - $Q_{ij} = \begin{cases} \lambda_i p_{ij} & if \;\; i \neq j \\ -\lambda_i & if \;\; i = j \end{cases}$

Daniel F. Silva,
silva@auburn.edu

Transition Diagrams for MC

- A graph where:
  - The nodes are each $s \in S$
  - The arcs are each $\lambda_i \times p_{ij} > 0 \; \forall i \neq j; i,j \in S$

- Example
  - $S = \{1,2\}$
  - $\lambda = \{1,10\}$
  - $P = \begin{pmatrix} 0.5 & 0.5 \\ 0.2 & 0.8 \end{pmatrix}$
  - $Q = \begin{pmatrix} -0.5 & 0.5 \\ 2 & -2 \end{pmatrix}$



Daniel F. Silva,
silva@auburn.edu

Measures of Performance

- $p_s(t)$: Probability of being in state s at time t
  - In vector form: $P(t)$
- π_s: Long-run average probability of being in state s
  - In vector form: π
- $MTTA$: Average time until absorption
  - Relevant when there is a set of "absorbing states"

Daniel F. Silva,
silva@auburn.edu

Transient Analysis

- The objective is to calculate $P(t)$ for some t

$$P(t) = a \cdot e^{Qt} = a \cdot \left(I + \sum_{n=1}^{\infty} \frac{(Qt)^n}{n!} \right)$$

Daniel F. Silva,
silva@auburn.edu

Steady-state analysis

- The objective is to compute π
- Solve the following system of equations

$$\pi Q = 0$$

$$\sum_{s \in S} \pi_s = 1$$

Daniel F. Silva,
silva@auburn.edu

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Mean Time To Absorption

- We can calculate MTTA as

$$MTTA = \sum_{s \in B} z_s$$

- Where A is the set of absorbent states, B are non-absorbent states and $A \cup B = S$

- And where

$$z_s = \int_0^\infty p_s(t)dt$$

Daniel F. Silva,
silva@auburn.edu

EXAMPLES OF MARKOV CHAIN MODELS FOR RELIABILITY, AVAILABILITY AND MAINTAINABILITY

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Two-state Component

- Consider a single component that fails at a constant hazard rate λ. Assume that repairs occur at a rate μ.
  - What is the probability that the component will be working in 1000 hours?
  - What is the long-run availability of the component?
  - What is the expected time to the first failure?

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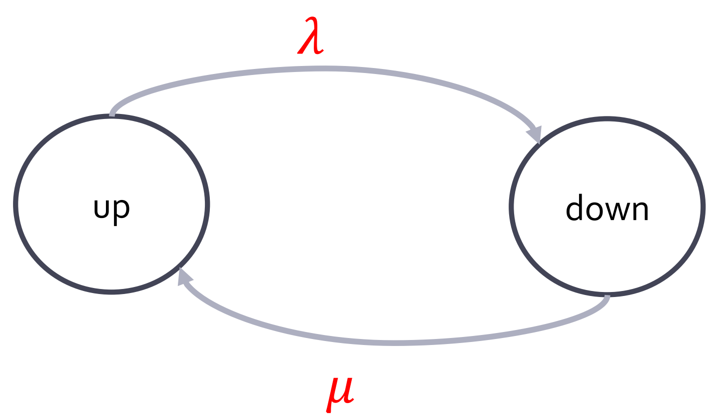
silva@auburn.edu

Two-state Component

- State space: $S = \{up, down\}$
- Transition diagram:



- Initial state: “up”

Daniel F. Silva,
silva@auburn.edu

Two-state Component

- The probability of being in each state after t is:

$$p_{up}(t) = \frac{\mu}{\lambda + \mu} + \frac{\lambda}{\lambda + \mu} e^{-(\lambda+\mu)t}$$

$$p_{down}(t) = \frac{\lambda}{\lambda + \mu} - \frac{\lambda}{\lambda + \mu} e^{-(\lambda+\mu)t}$$

- Therefore:

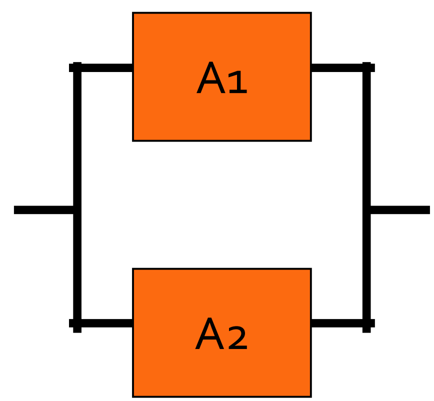
$$\lim_{t\to\infty} p_{up}(t) = \pi_{up} = \frac{\mu}{\lambda + \mu}$$

$$\lim_{t\to\infty} p_{down}(t) = \pi_{down} = \frac{\lambda}{\lambda + \mu}$$

Daniel F. Silva,
silva@auburn.edu

N parallel components with shared repairs

- Consider a system with N identical components in parallel, which fail at a constant rate λ. Assume that repairs occur one at a time at a rate μ, per repair. There is only one repair resource
  - What is the long-run availability of the system?
  - On average how many components are operational?



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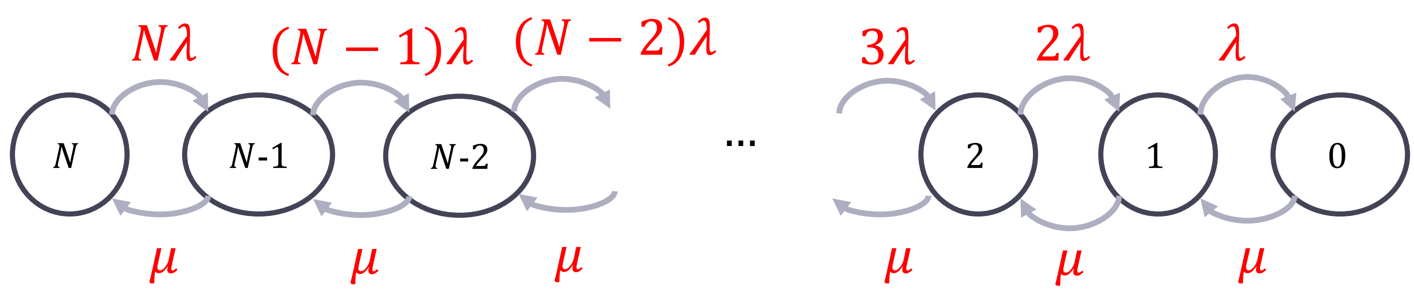
silva@auburn.edu

N parallel components with shared repairs

- State = Number of working components
- State space: $S = \{0,1,2,\ldots,N\}$
- Transition diagram:



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silva@auburn.edu

N parallel components with shared repairs

- Balance equations

$$N\lambda\pi_N = \mu\pi_{N-1}$$
$$\dots$$
$$2\lambda\pi_2 = \mu\pi_1$$
$$\lambda\pi_1 = \mu\pi_0$$

- Normalization:

$$\sum_{n=1}^{N} \pi_n = 1$$

- Then,

$$Availability = 1 - \pi_0 = 1 - \left(\sum_{n=0}^{N} \frac{\mu^n}{n!\,\lambda^n}\right)^{-1}$$

Daniel F. Silva,
silva@auburn.edu

N parallel components with shared repairs

- For example for

$$\lambda = 0.01hr^{-1}, \qquad \mu = 0.1hr^{-1}, \qquad N = 5$$

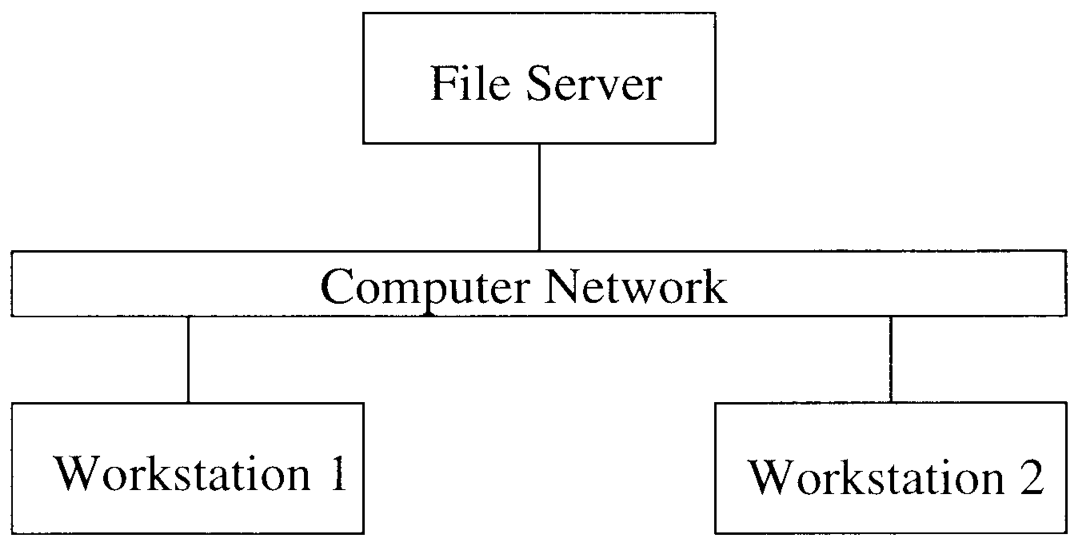
$$Availability = 1 - \pi_0 = 1 - \left(\sum_{n=0}^{N} \frac{\mu^n}{n!\,\lambda^n}\right)^{-1} = 99.932\%$$

$$E[Operational] = \sum_{n=1}^{N} n \times \frac{\mu^n}{n!\,\lambda^n} \pi_0 = 4.36$$

Daniel F. Silva,
silva@auburn.edu

Workstation-Fileserver

- Consider a system with 2 identical workstations and one fileserver, connected by a network.
- The system is operational as long as:
  - At least 1 workstation is up
  - The fileserver is up



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silva@auburn.edu

Workstation-Fileserver

- Assuming exponentially distributed times to failure
  - λ_w : failure rate of workstation
  - λ_f: failure rate of file-server
- Assume that components are repairable
  - μ_w: repair rate of workstation
  - μ_f: repair rate of file-server
- Shares repairs. File-server has priority for repair over workstations

Daniel F. Silva,
silva@auburn.edu

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Workstation-Fileserver

- State = (Number of operational workstations, number of operational fileservers)
- $S = \{(2,1), (1,1), (0,1), (2,0), (1,0), (0,0)\}$

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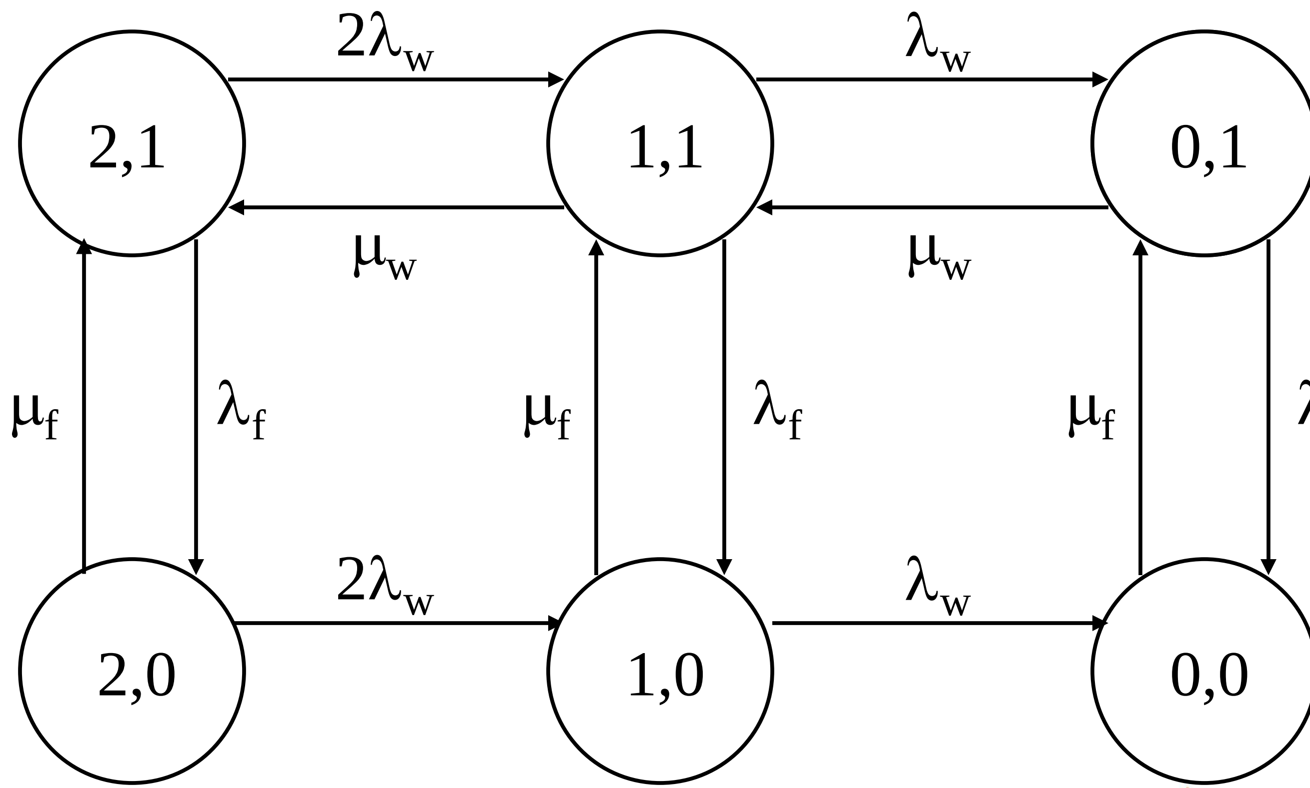
silva@auburn.edu

Workstation-Fileserver

- Transition diagram:



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silva@auburn.edu

Workstation-Fileserver

- Long run average Availability

$$Availability = \pi_{2,1} + \pi_{1,1}$$

- Example:

$$\lambda_w = 0.0001\, h^{-1}, \lambda_f = 0.00005\, h^{-1}, \mu_w = 1.0\, h^{-1}, \mu_f = 0.5\, h^{-1}$$

- Then

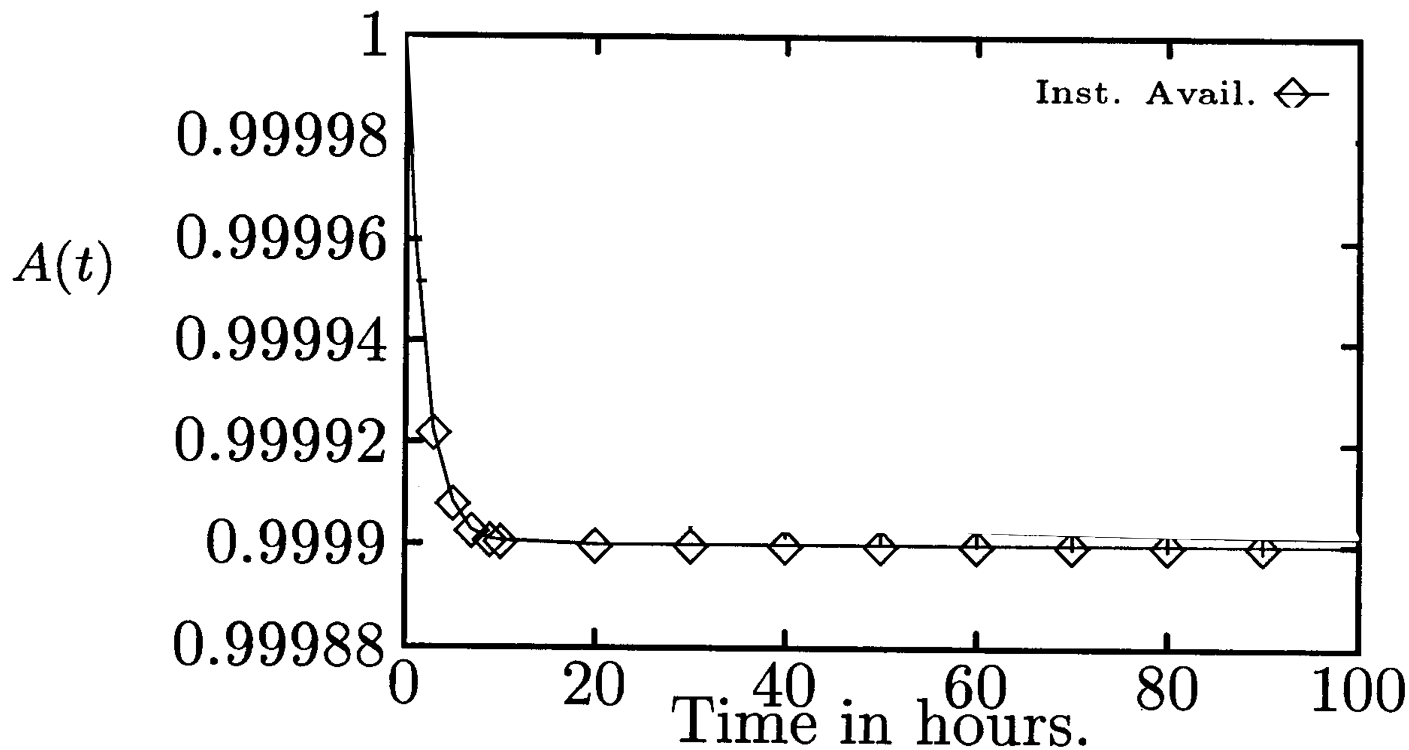
$$Availability = 0.9999$$

Daniel F. Silva,
silva@auburn.edu

Workstation-Fileserver

- Instantaneous availability
- $A(t) = p_{2,1}(t) + p_{1,1}(t)$

$$\lambda_w = 0.0001\,h^{-1}, \lambda_f = 0.00005\,h^{-1}, \mu_w = 1.0\,h^{-1}, \mu_f = 0.5\,h^{-1}$$



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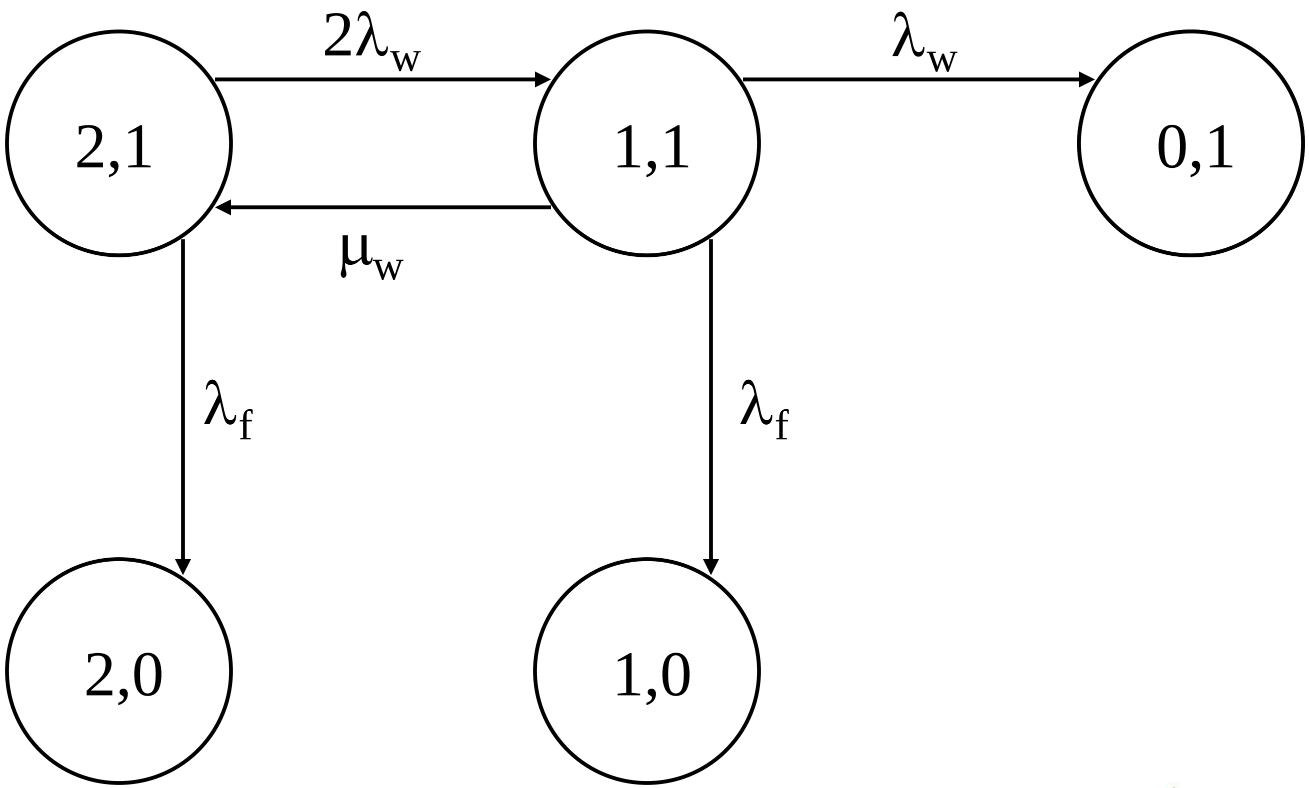
Workstation-Fileserver

- Calculate time until system failure



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silva@auburn.edu

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- Let

$$\lambda_w = 0.0001\,h^{-1}, \lambda_f = 0.00005\,h^{-1}, \mu_w = 1.0\,h^{-1}, \mu_f = 0.5\,h^{-1}$$

- $MTTF = z_{2,1} + z_{11}$

- We can solve $z_{2,1}, z_{1,1}$ numerically using MATLAB as:
  - $z_{2,1} = \int_0^\infty p_{2,1}(t)dt\,, \qquad z_{1,1} = \int_0^\infty p_{1,1}(t)dt$

- Then $MTTF\ =\ 19992\ hours$

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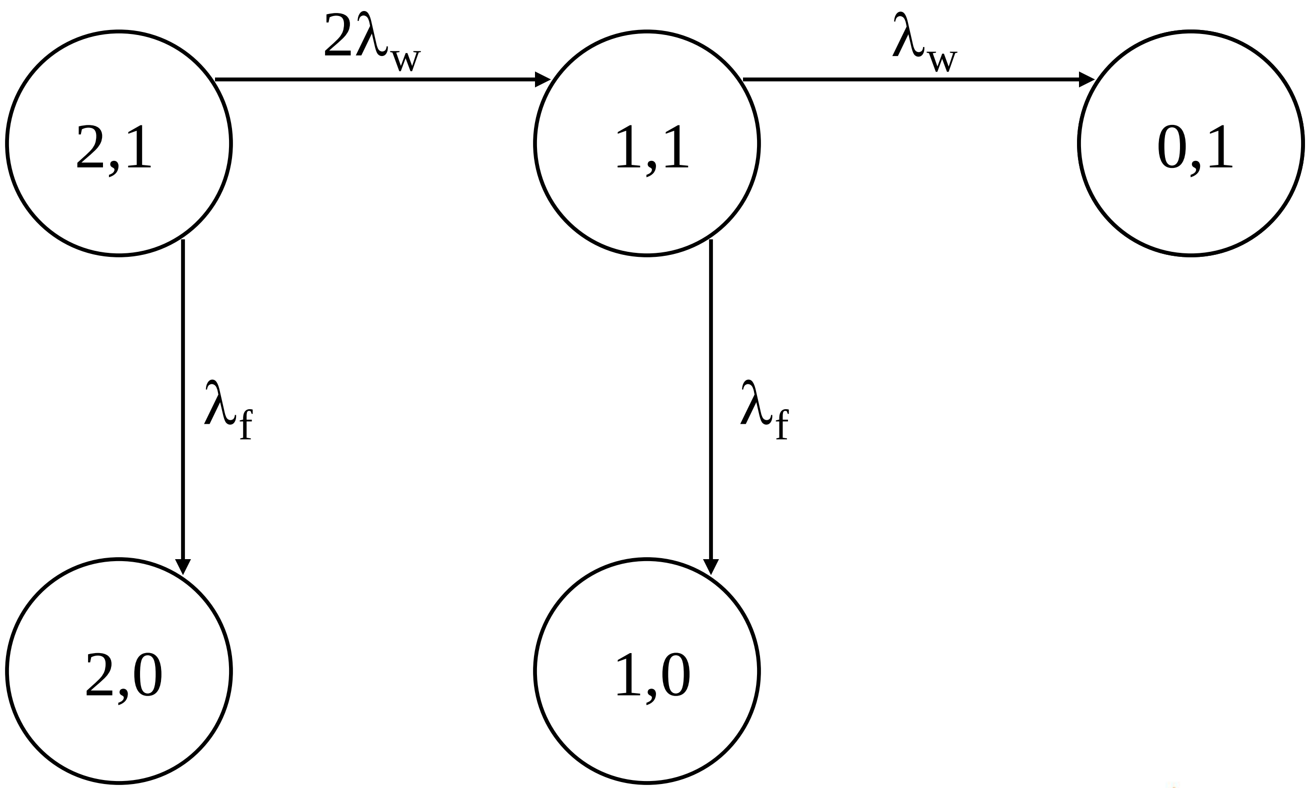
silva@auburn.edu

Workstation-Fileserver

- Suppose workstations cannot be repaired



Daniel F. Silva,
silva@auburn.edu

Workstation-Fileserver

- Let

$$\lambda_w = 0.0001\, h^{-1}, \lambda_f = 0.00005\, h^{-1}, \mu_w = 1.0\, h^{-1}, \mu_f = 0.5\, h^{-1}$$

- $MTTF = z_{2,1} + z_{11}$

- We can solve $z_{2,1}, z_{1,1}$ numerically using MATLAB as:
  - $z_{2,1} = \int_0^\infty p_{2,1}(t)dt\,, \qquad z_{1,1} = \int_0^\infty p_{1,1}(t)dt$

- Then $MTTF\ = 9993\ hours$

Daniel F. Silva,
silva@auburn.edu

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EXTENSIONS AND MORE COMPLEX MODELS

Infinite Markov Chains

- Suppose that your system consists of infinitely many states.
- Example: The state represents the number of components awaiting repair, from an infinite pool.
- Some infinite-state MCs are well understood
  - Traditional queues M/M/1, M/M/c, etc.
  - So-called quasi-birth-death processes.
- These kinds of models can be solved by
  - Analytical methods (for queues)
  - Matrix-Analytic methods (for quasi-birth-death processes).
  - Fluid approximations to a continuous system

Daniel F. Silva,
silva@auburn.edu

Phase-type Distributions

- Suppose that not all times are exponential
- Example: Repair times a “nearly” deterministic.
- PH-type distributions allow us to model the holding times at some states as other distributions.
- PH-type distributions use a **new MC** to model a **single transition.**
- Some distributions that can be well approximated by PH-type are:
    - Erlang
    - Deterministic
    - Hyper- and Hypo-exponential

Markov Decision Process (MDP)

- Suppose you can make a decision at each transition.
- Example: Choose which component gets a repaired.
- MDPs add an Action Space A to the definition of a MC.
- Now the problem is not just determining performance.
- We choose a **policy** (the action to take in each state).
- The objective is to optimize a certain measure of performance. Like
  - Maximize time to failure
  - Maximize average availability
  - Maximize number of repairs

Daniel F. Silva,
silva@auburn.edu

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COMPUTATIONAL TOOLS FOR MARKOV MODELS

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Software for Markov Chains

- SMCSolver
- Butools
- SHARPE
- MKV
- **jMarkov**
  - Available at www.jmarkov.org

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Daniel F. Silva,
silva@auburn.edu

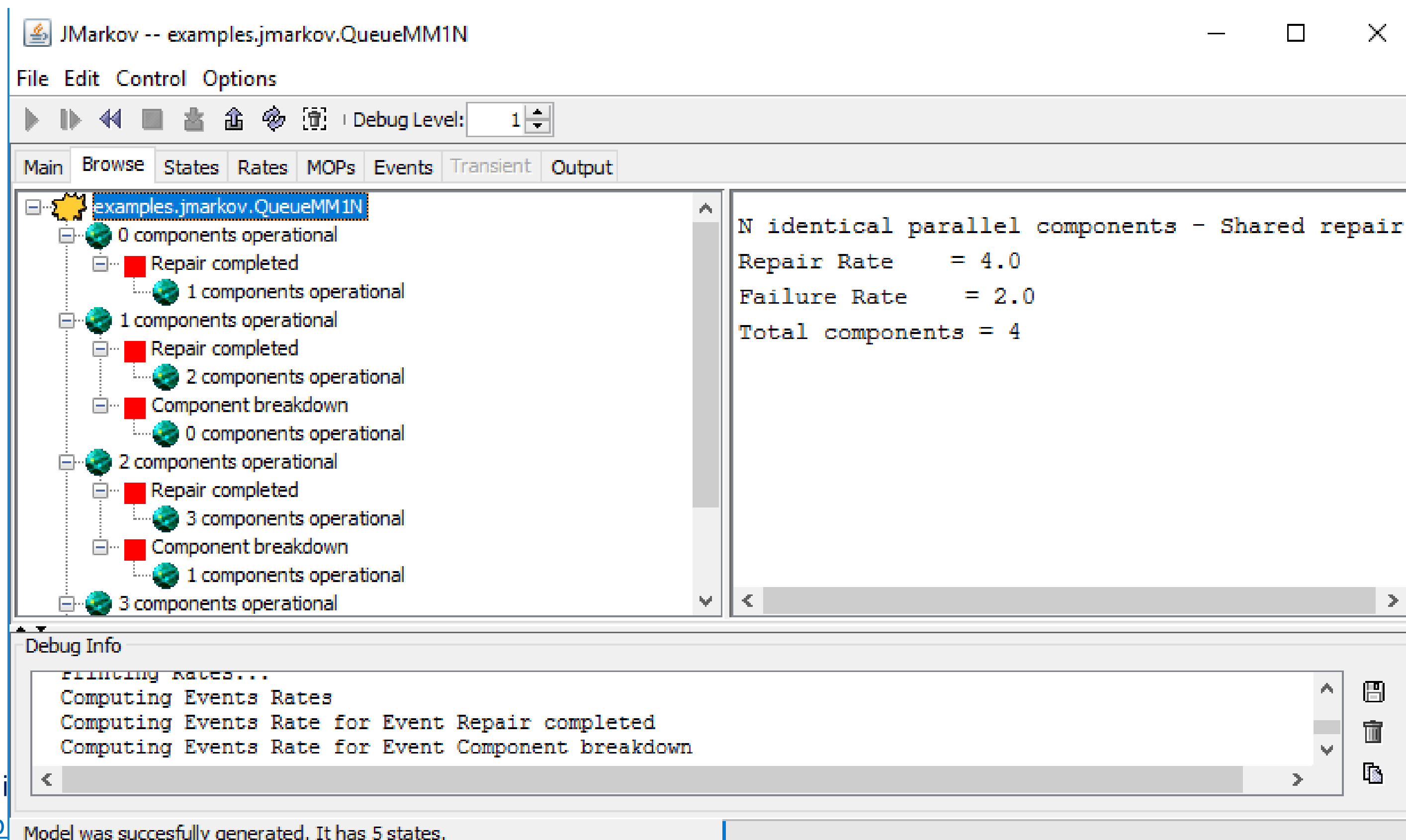
jMarkov

- Object-oriented framework for Markov models
- Designed for modeling and solving
- Coded in Java
- Can be imported into other programs
- Consists of four modules:
  - **Core module**: General finite MCs
  - **jQBD**: Highly structured infinite MCs
  - **jPhase**: Structured MCs with phase-type times
  - **jMDP**: Markov Decision Processes

Daniel F. Silva,
silva@auburn.edu

jMarkov – Core module

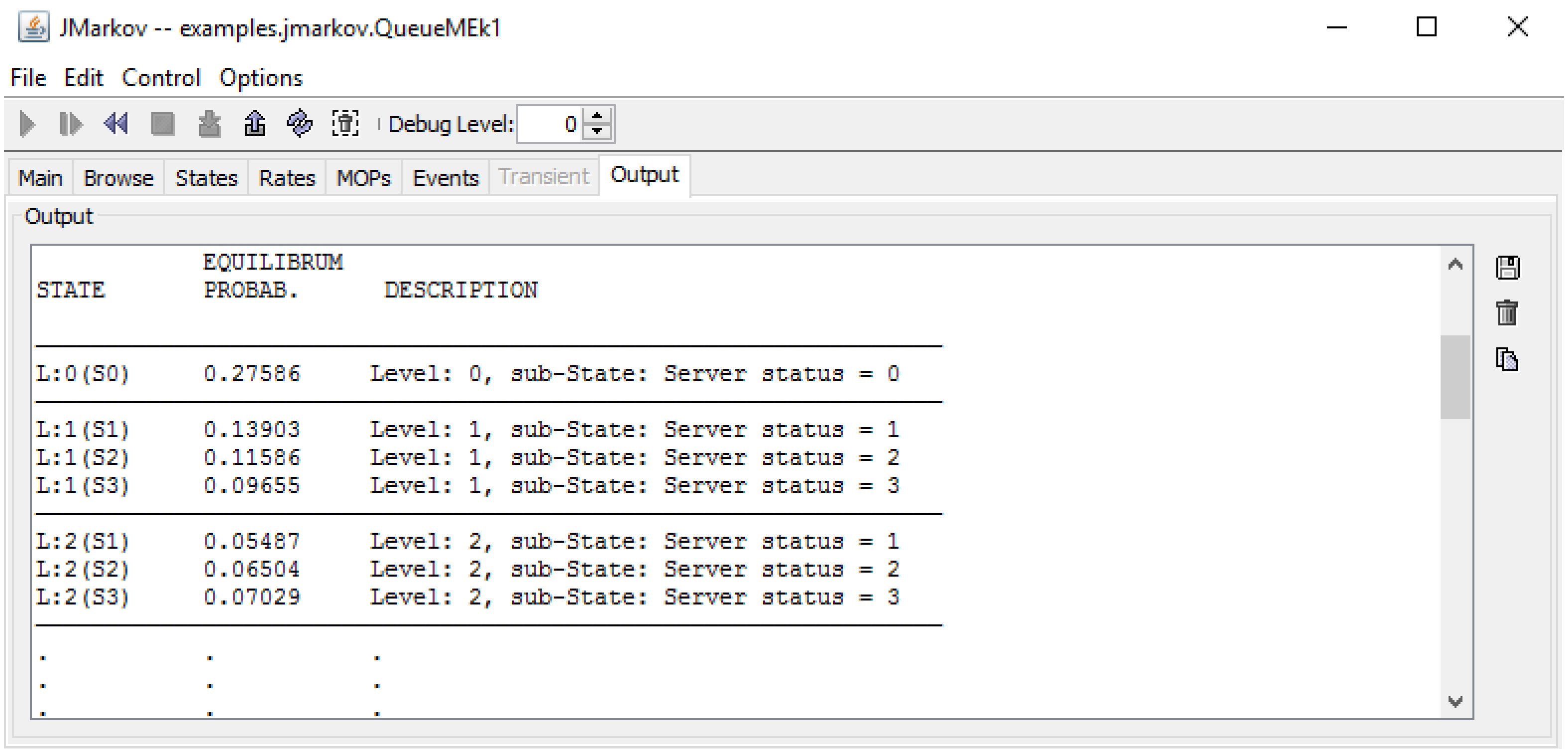
- Builds MC from simple rules.
- Handles finite discrete and continuous-time MCs.
- Calculates π and $P(t)$ and expected rewards.



Daniel F. Si
silva@aub

jQBD

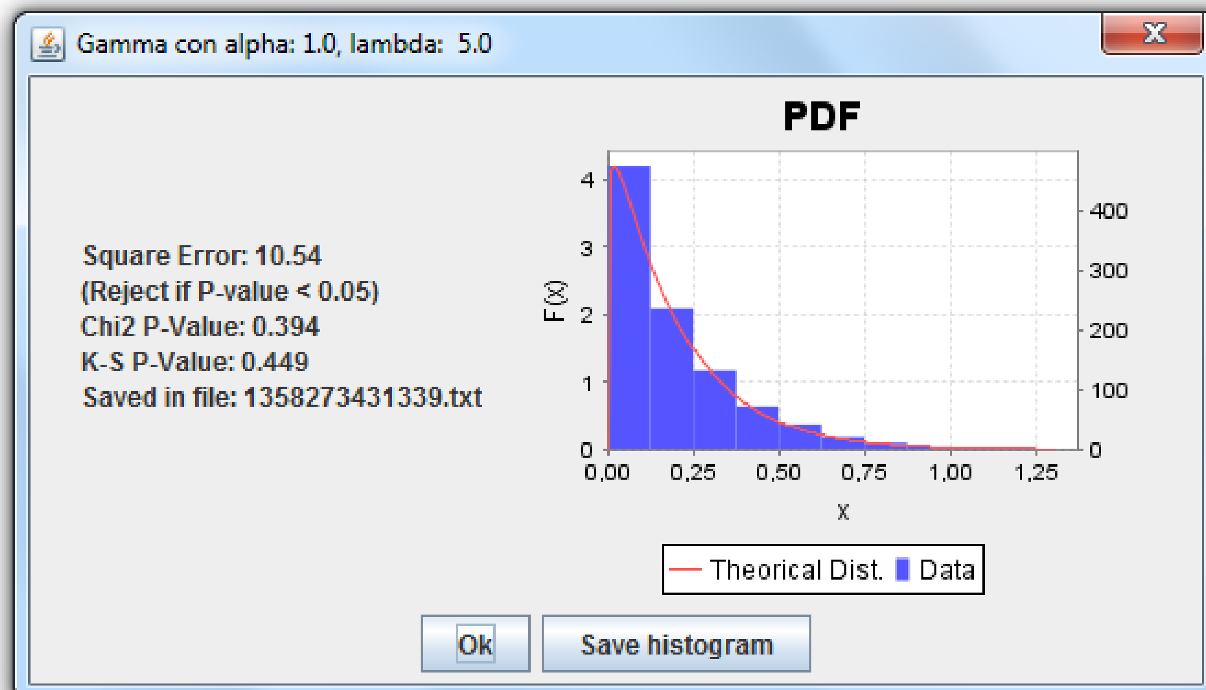
- Only needs specification of the repeating structure.
- Calculates expected rewards.



Daniel F. Silva,
silva@auburn.edu

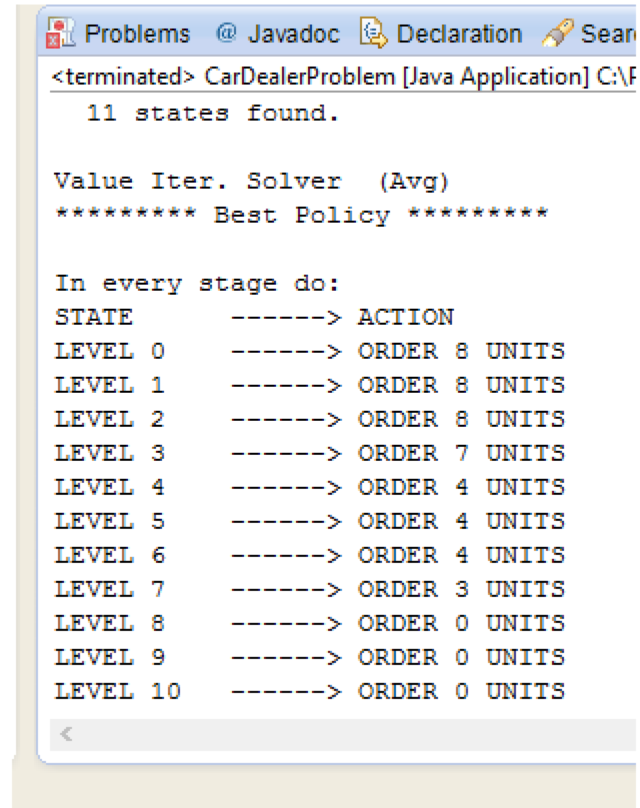
jPhase

- Allows specification of PH-type random variables
- Permits including PH-type variables in MC models
- Includes fitting capabilities for estimating PH parameters
- Generates PH-type random variates for simulations



jMDP

- Builds MDP models from simple rules
- Supports finite and infinite horizon, discrete and continuous time MDPs
- Calculates optimal deterministic policies for total, average and discounted cost criteria
- Solvers calculate model parameters on-the-fly




```
 11 states found.

Value Iter. Solver  (Avg)
********* Best Policy *********

In every stage do:
STATE       ------> ACTION
LEVEL 0     ------> ORDER 8 UNITS
LEVEL 1     ------> ORDER 8 UNITS
LEVEL 2     ------> ORDER 8 UNITS
LEVEL 3     ------> ORDER 7 UNITS
LEVEL 4     ------> ORDER 4 UNITS
LEVEL 5     ------> ORDER 4 UNITS
LEVEL 6     ------> ORDER 4 UNITS
LEVEL 7     ------> ORDER 3 UNITS
LEVEL 8     ------> ORDER 0 UNITS
LEVEL 9     ------> ORDER 0 UNITS
LEVEL 10    ------> ORDER 0 UNITS
```


Daniel F. Silva,
silva@auburn.edu

INTERNATIONAL STANDARDS FOR APPLYING FOR MARKOV MODELS IN RAM

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IEC Standards

- The International Electrotechnical Commission (IEC) has 2 standards concerning Markov modeling techniques:
  - **IEC 61165:** Application of Markov techniques
  - **IEC 61508:** Functional safety of electrical/ electronic/ programmable electronic safety-related systems

Daniel F. Silva,
silva@auburn.edu

IEC 61165

- Provides guidance on the application of Markov techniques to model and analyze a system and estimate reliability, availability, maintainability and safety measures.
  - Gives an overview of available methods.
  - Reviews the relative merits of each method and their applicability.

Daniel F. Silva,
silva@auburn.edu

IEC 61508

- Part 1: General requirements
- Part 2: Requirements for electrical/ electronic/ programmable safety-related systems
- Part 3: Software requirements
- Part 4: Definitions and abbreviations
- Part 5: Examples of methods for the determination of safety integrity levels
- Part 6: Guidelines on the application of IEC 61508-2 and -3
- Part 7: Overview of techniques and measures

Daniel F. Silva,
silva@auburn.edu

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SUMMARY & CONCLUSION

Conclusions

- Markov model are a powerful mathematical modeling technique.
- Markov Chains can be used in many applications of Reliability, Availability and Maintainability
- Markov models can provide exact analytical solutions to small problems.
- More elaborate Markov models can incorporate decision-making, non-exponential distributions and other extensions.
- Computational tools, such as jMarkov, can be used to solve large scale Markov models easily.

Daniel F. Silva,
silva@auburn.edu

THANK YOU

Contact:
Daniel F. Silva
silva@auburn.edu
www.jmarkov.org

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