The Reliability Function

HSV RAM XIV

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Probability Theory

A **Probability Space** is a triple (Ω, E, P), such that

1. Ω is a non-empty set of all possible outcomes called the **Sample Space**
2. E is a collection of subsets of Ω called **Events**
   - E must be closed under complements and unions
3. $P: E \to [0, 1]$ is called a **Probability Measure**
   - $P(\Omega) = 1$
   - $P(\bigcup A_i) = \sum P(A_i)$, where $A_i \in E$ and A_i are pairwise disjoint*
4. *Additional rigorous details that are important to Mathematicians*

*Pair of dice example: $P(rolling\ a\ 4) = P([1, 3]) + P([3, 1]) + P([2, 2])$

Random Variables

A **Random Variable** is a function X that maps outcomes of an experiment to numerical quantities.

Therefore, we can express mathematically, the probability that the outcome to an experiment is mapped by X to a quantity within an interval

$$P(a \leq X \leq b) \stackrel{\text{def}}{=} P([a, b])$$

Probability Distributions ($\Omega = \mathbb{R}_{\geq 0}$ case)

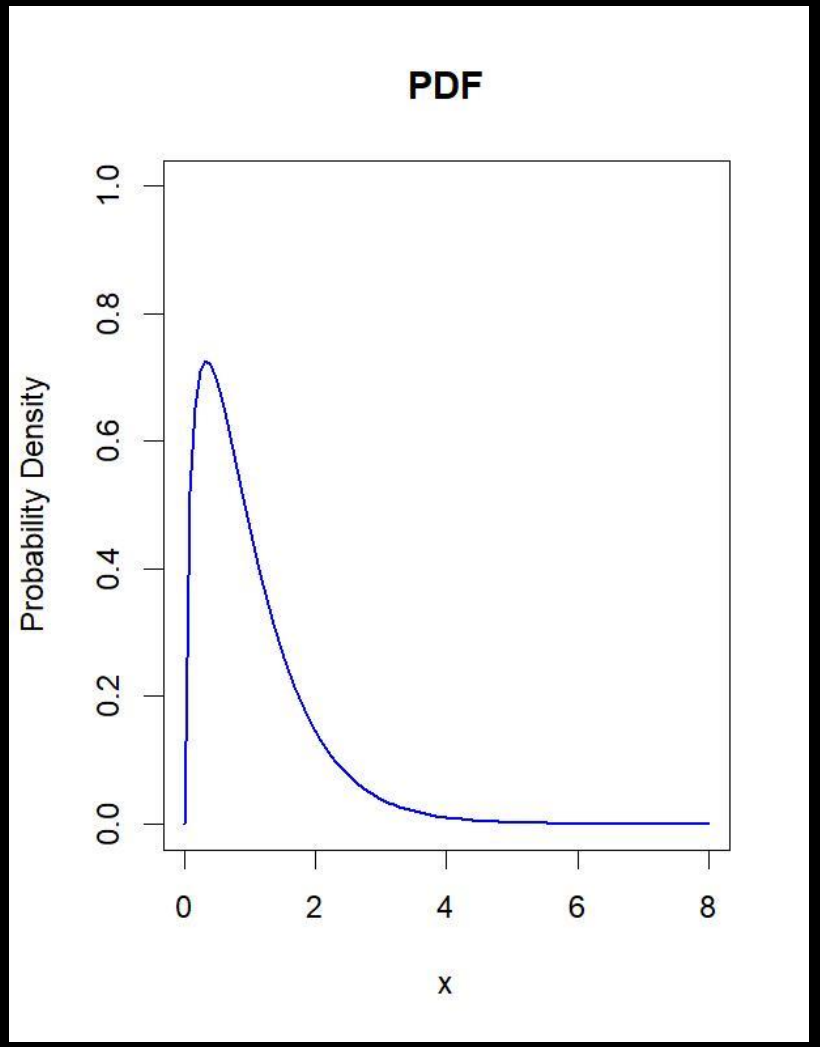
A **Probability Distribution** is a mathematical model used to construct probability measures. **Probability Density Functions** are one such characterization for continuous random variables. They are analogous to **Histograms**, and they have useful geometric interpretations.

A **Probability Density Function** is a density function $f(x)$ defined on $\mathbb{R}_{\geq 0}$ such that,

$$P(\mathbb{R}_{\geq 0}) = \int_0^\infty f(x)dx = 1$$

And,

$$P([a,b]) = \int_a^b f(x)dx$$



The Reliability Function (Definitions)

The **Cumulative Distribution Function** $F(x)$ is defined as

$$F(x) \stackrel{\text{def}}{=} \int_0^x f(z)dz = P(X \leq x)$$

The **Inverse Cumulative Distribution Function** $F^{-1}(p)$ is defined at p to be the value x such that $F(x) = p$

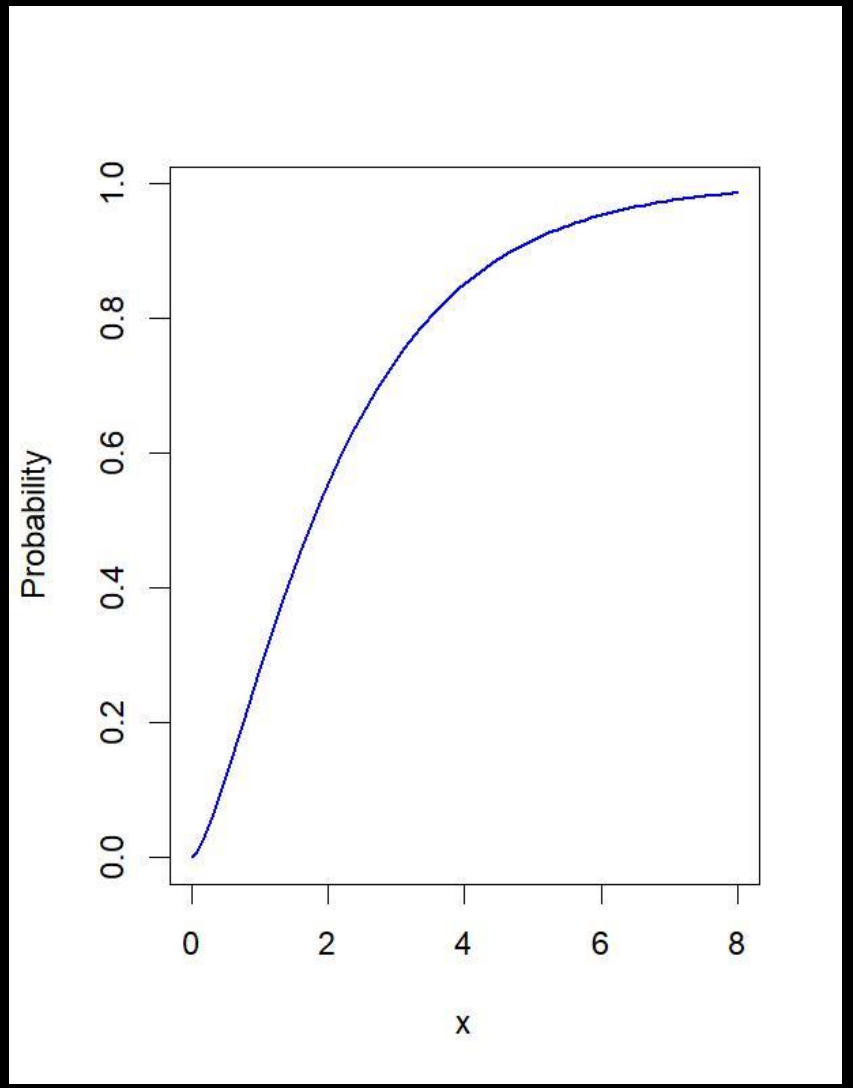
The **Reliability Function** or **Survival Function** is defined as the complement of the Cumulative Distribution Function:

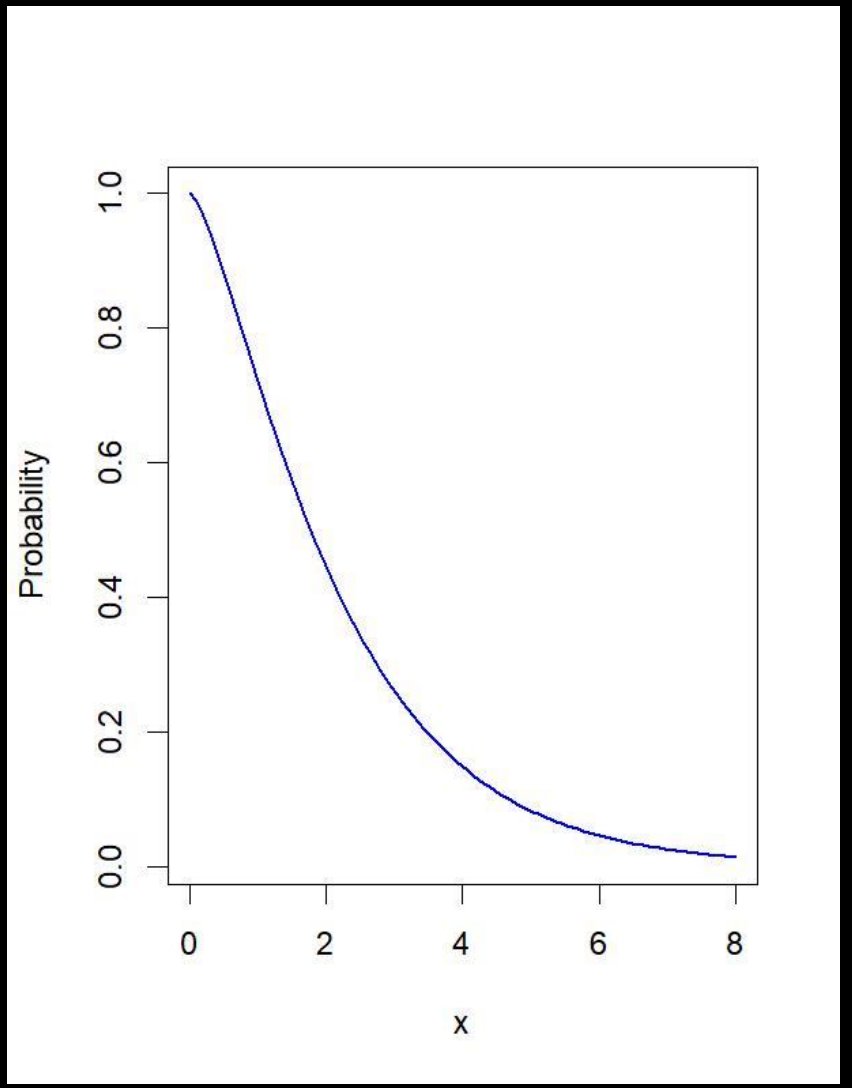
$$S(x) \stackrel{\text{def}}{=} 1 - F(x) = P(X > x)$$

The Reliability Function (Quiz 1)

$$F(x) = P(X \leq x)$$

$$S(x) = P(X > x)$$





Example Distributions (Quiz 2)

The Exponential Distribution (λ)

$$f(x) = \lambda e^{-\lambda x}$$

$$F(x) = 1 - e^{-\lambda x}$$

$$S(x) = e^{-\lambda x}$$

The Weibull Distribution (β, η)

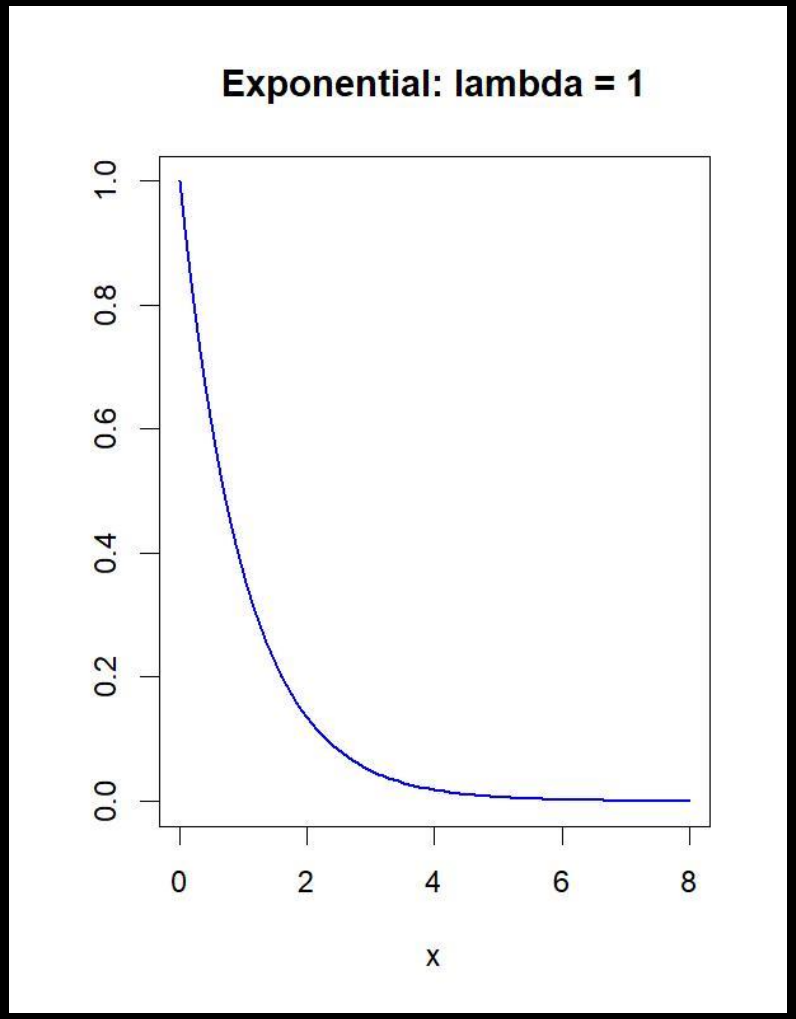
$$f(x) = \frac{\beta}{\eta}\left(\frac{x}{\eta}\right)^{\beta-1} e^{-(x/\eta)^{\beta}}$$

$$F(x) = 1 - e^{-(x/\eta)^{\beta}}$$

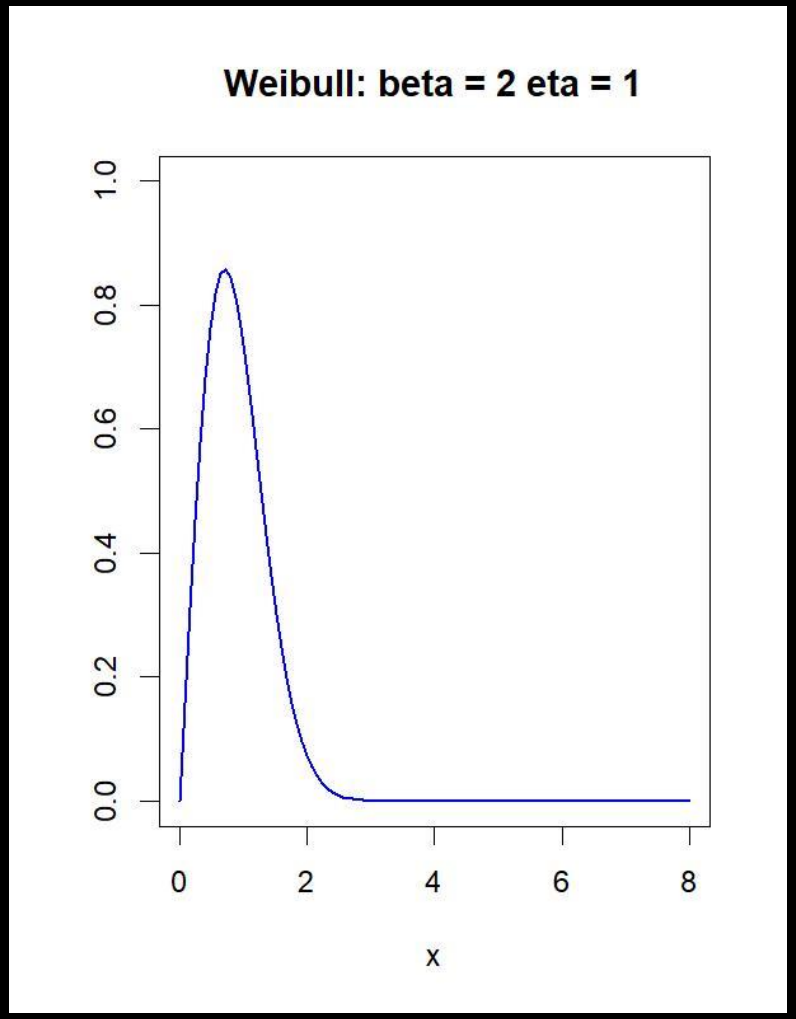
$$S(x) = e^{-(x/\eta)^{\beta}}$$

Example Distributions (Quiz 3)

The Exponential Distribution (λ)



The Weibull Distribution (β, η)



Fatigue Life Example (Purpose)

Material Fatigue can be thought of as cumulative yet difficult to detect damage to material through cyclical loading. A reasonable and potentially useful application of the reliability function would be to model the fatigue life of rivets.

Recall an earlier problem*:

> *A small boat manufacturer wants to know the probability that a rivet fails, creating a hole that could sink their boats.*
>
> *Each boat uses 600 rivets, and the manufacturer has already built 6 boats with no occurrences of this failure.*

Probability that Boat Sinks due to Rivet Failure?

*Expectation vs. Reality: The Sunrise Problem Applied to Probabilistic Risk Assessment, Joseph Osowski

Fatigue Life Example (Definitions)

- **Rivet failure** is defined to be crack growth through to complete fracture
- Define the **end of fatigue life** to be detectible crack initiation.
- Boat rivet fatigue life can be decomposed into units called sea years. One **sea year** is defined to be the amount of fatigue damage incurred by a rivet from exposure to cyclical loading equivalent to that of a single year at sea.

Given a set of fatigue life spectral data from rivet testing and boat trip loads data, it is possible to fit a probability distribution that can be used to model the reliability of rivets against crack initiation.

Fatigue Life Example (Problem Statement)

100 rivets were repeatedly tested under a cyclical loading profile until all rivets experienced crack initiation. After each set of 100 test runs, the rivets were inspected for cracks. Analysis has also determined that a single test load profile is equivalent to 1/3 of a sea year.

- Notional Test Data:
- 15 rivets cracked between 600 and 700 test runs
- 80 rivets cracked between 700 and 800 test runs
- 5 rivets cracked between 900 and 1000 test runs

Boat level requirements state that each rivet must have a reliability greater than 0.99995 throughout qualified service life

Requirements can be verified by fitting this data to a Weibull distribution and deriving a service life specification

Fatigue Life Begins → Crack Initiation → Rivet Failure → Boat Sinks

Fatigue Life Example (Weibull MLE Solution)

Maximize L for β and η:

$$L(\beta,\eta) = [S(600) - S(700)]^{15} \cdot [S(700) - S(800)]^{80} \cdot [S(900) - S(1000)]^{5}$$

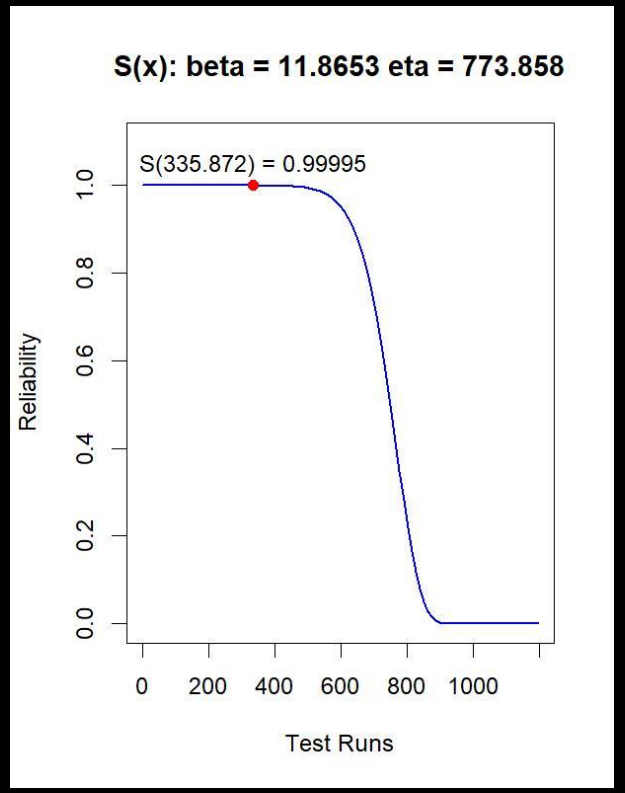
Solution:

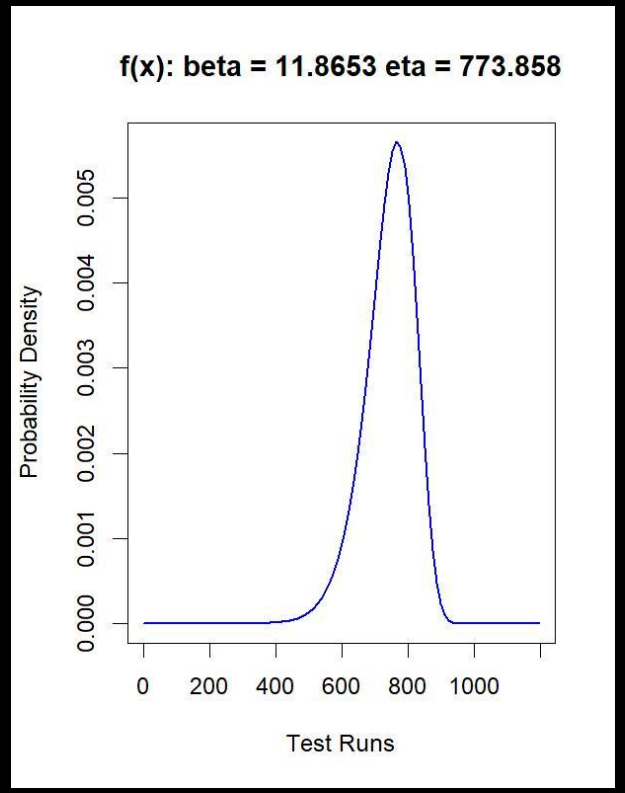
$\beta = 11.8653$

$\eta = 773.8583$

Rivet reliability requirement verification remains valid for a service life of 110 years at sea

- $S(210) = 0.9999998$
- 1 in 10,000 chance of crack initiation during a 70-year service life at sea





Questions?

Backup

Glossary

Cumulative Distribution Function (CDF)	The CDF of a real-valued random variable X, evaluated at x, is the probability that X will take a value less than or equal to x.
Event	An event is a set of outcomes of an experiment or a subset of the sample space.
Experiment	An experiment is any procedure that can be infinitely repeated and has a well-defined set of possible outcomes.
Fatigue	Cyclic load induced flaws and damage to material that can ultimately lead to crack initiation.
Histogram	A histogram is an approximate representation of the distribution of numerical data.
Likelihood Function	The likelihood function describes the joint probability of the observed data as a function of the parameters of a given statistical model.
Maximum Likelihood Estimate (MLE)	MLE is a method of estimating the parameters of an assumed probability distribution, given observed data.
Numerical Method	A numerical method is a mathematical tool designed to solve numerical problems.
Probability Density Function (PDF)	The PDF of a continuous random variable is used to specify the probability of the random variable falling within a particular range of values.
Probability Distribution	A Probability Distribution is a mathematical model used to construct probability measures.
Probability Measure	In mathematics, the concept of a measure is a generalization and formalization of geometrical measures (distance/length, area, volume) and other common notions, such as mass and probability of events. Probability measures must satisfy all the general measure properties as well as the property that a probability measure must assign value 1 to the entire probability space.
Probability Space	A probability space is a mathematical formalism of a random process or experiment.
$\mathbb{R}_{\geq 0}$	The non-negative real numbers.
Random Variable	A Random Variable is a function X that maps outcomes of an experiment to numerical quantities.
Reliability	The probability that a product, system, or service will perform its intended function adequately for a specified duration or will operate in a defined environment without failure.
Reliability/Survival Function	The reliability function is theoretically defined as the function of t that returns the probability of success at time t.
Sample Space	The sample space of an experiment or random trial is the set of all possible outcomes or results of that experiment.