Pragmatic Weibull Analysis

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Prerequisites

- Calculus
- Linear Algebra
- Probability Theory
- Mathematical, Critical and Analytical Thinking

$$\frac{e^{-x^2/2}}{\sqrt{2\pi}}$$

$$\frac{e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}}{\sigma\sqrt{2\pi}}$$

Take aways

- Practical knowledge of Weibull analysis
- Self-reliant confidence in fitting curves

$$e^{-x}$$

$$\lambda e^{-\lambda x}$$

$$\frac{\beta}{\eta}\left(\frac{x}{\eta}\right)^{\beta-1} e^{-(x/\eta)^{\beta}}$$

History

- Fréchet (fray-SHAY) 1927 $\left(F = \frac{1}{W}; reciprical\ of\ the\ Weibull\right)$
  - While investigating the limiting behavior of the maximum of independent and identically distributed (i.i.d.) random variables, Fréchet showed that under certain conditions, the distribution of the maximum converges to:

$$f(x) = \frac{\alpha}{s}\left(\frac{x-m}{s}\right)^{-1-\alpha} e^{-\left(\frac{x-m}{s}\right)^{-\alpha}}$$

- Rammler (RAHM-ler) 1933 $(Equivalent\ to\ the\ Weibull)$
  - The Rammler distribution assumes that an exponential relationship exists between the particle diameter (d) and the mass fraction of particles with diameter greater than d.

$$Y_d \stackrel{\text{def}}{=} e^{-(d/\bar{d})^n}$$

- Weibull (VY-bull) 1939 $\left(W = \frac{1}{F}; reciprical\ of\ the\ Fréchet\right)$

Probability Distributions ($\Omega = \mathbb{R}_{\geq 0}$ case)

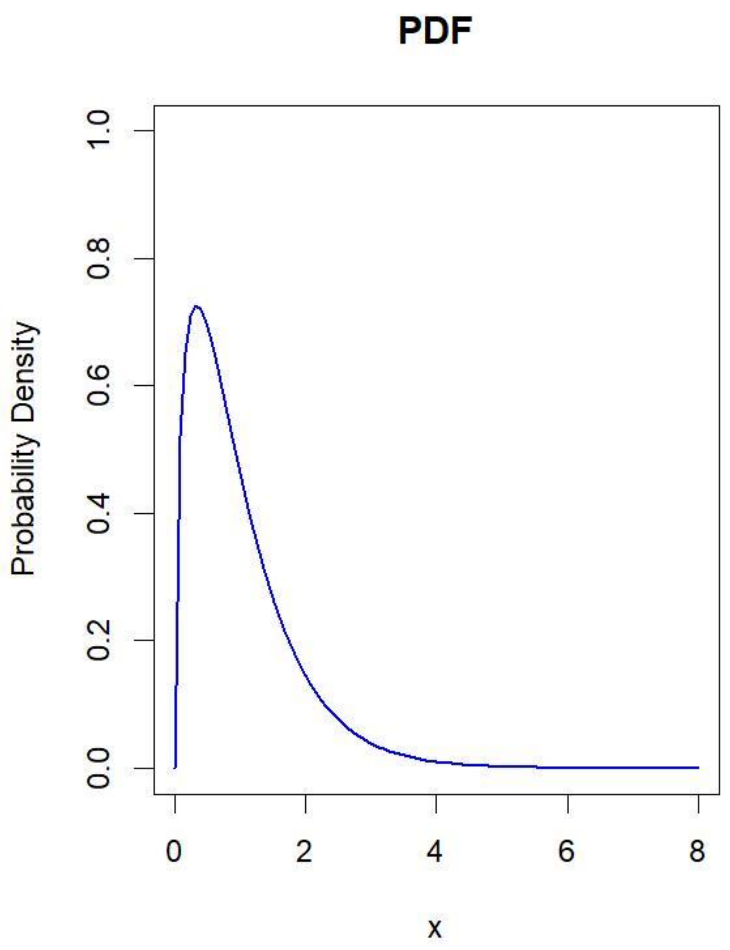
A **Probability Distribution** is a mathematical model used to construct probability measures. **Probability Density Functions** are one such characterization for continuous random variables. They are analogous to **Histograms**, and they have useful geometric interpretations.

A **Probability Density Function** is a density function $f(x)$ defined on $\mathbb{R}_{\geq 0}$ such that,

$$P(\mathbb{R}_{\geq 0}) = \int_0^\infty f(x)dx = 1$$

And,

$$P([a,b]) = \int_a^b f(x)dx$$



Definitions ($\Omega = \mathbb{R}_{\geq 0}$ case)

The **Cumulative Distribution Function** $F(x)$ is defined as

$$F(x) \stackrel{\text{def}}{=} \int_0^x f(z)dz = P(X \leq x)$$

The **Inverse Cumulative Distribution Function** $F^{-1}(p)$ is defined at p to be the value x such that $F(x) = p$

The **Reliability Function** or **Survival Function** is defined as the complement of the Cumulative Distribution Function:

$$S(x) \stackrel{\text{def}}{=} 1 - F(x) = P(X > x)$$

Weibull Distribution Overview

$$f(x) = \frac{\beta}{\eta}\left(\frac{x}{\eta}\right)^{\beta-1} e^{-(x/\eta)^{\beta}}$$

$$F(x) \stackrel{\text{def}}{=} \int_0^x f(t)dt = 1 - e^{-(x/\eta)^{\beta}}$$

Example Weibull plots for beta = 4, eta = 3

- beta represents the acceleration of failure rate
- eta represents characteristic life independent of beta

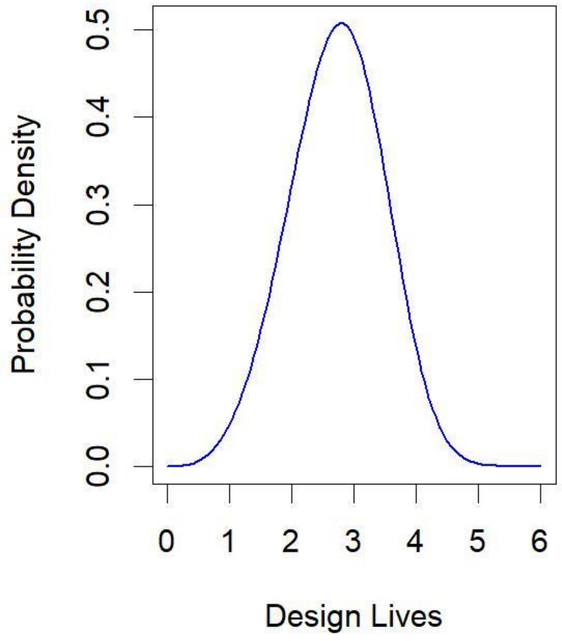
Note1: $F(\eta) \cong 0.63 \; and \; S(\eta) \cong 0.37$

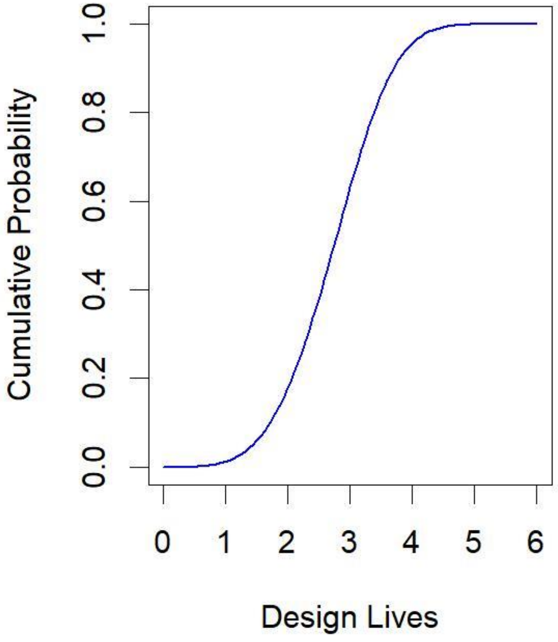
Note2: $For \; \beta > 1, \; h(x) \; increases \; over \; "time"$

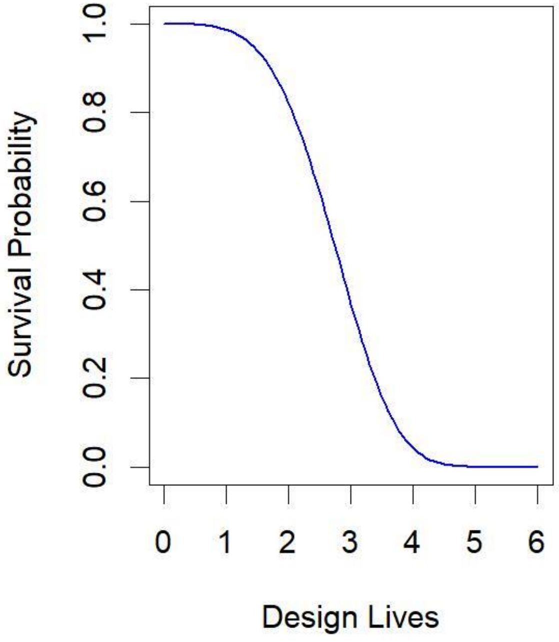
f(x) ← "Failure Space" → **F(x)**

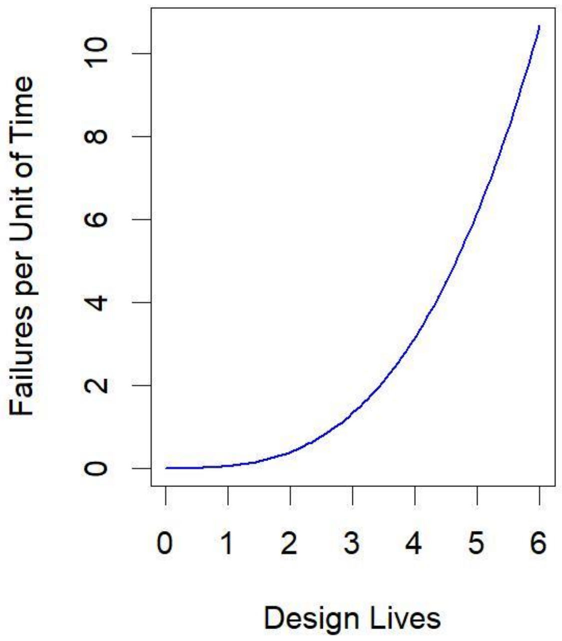
S(x) = 1 - F(x)

h(x) = f(x) / S(x)

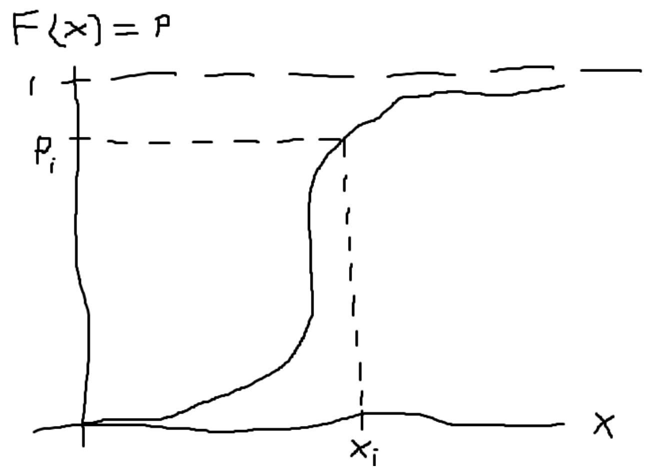








Notional Data



Generate evolving data sets from an arbitrary Weibull Distribution:

1. Select *representative* parameters
2. Define the inverse cumulative distribution
3. Sample random numbers p_i uniformly from $[0,1]$
4. Compute $F^{-1}(p_i)$

beta =	3
eta =	6

$$f(x) = \frac{\beta}{\eta}\left(\frac{x}{\eta}\right)^{\beta-1} e^{-(x/\eta)^{\beta}}$$

$$F(x) = p$$

$$F(x) = 1 - e^{-(x/\eta)^{\beta}}$$

$$F^{-1}(p) = x$$

$$F^{-1}(p) = \eta(-\ln(1-p))^{\frac{1}{\beta}}$$

x
4.03
7.91
5.35

x
4.03
7.91
5.35
3.64
5.48
4.07

x	
x_1	4.03
x_2	7.91
x_3	5.35
x_4	3.64
x_5	5.48
x_6	4.07
x_7	4
x_8	7.8
x_9	1.92

Apply curve fitting strategies:

1. Fit curves (illustrated on subsequent slides) to each of the three evolving data sets
2. Compare estimated parameters to the selected *representative* parameters
3. Visually examine the distribution of the samples against the “fitted” curves
4. Analyze what each strategy is doing at key steps in their respective procedures

Summary of Two Strategies

- Maximum Likelihood Estimate (MLE)
  - Solves for the density curve that **maximizes** the likelihood of the data given the curve
  1. Begins with defining the Likelihood Function
  2. Follows with taking the log of the Likelihood and expressing eta as a function of beta
  3. Ends with numerically "maximizing" the log-likelihood function
  - Pros: automatically accommodates censored data sets
  - Cons: solutions do not always exist
- Median Rank Regression (MRR)
  - Solves for the cumulative curve that **minimizes** the "errors" between the data and curve
  1. Begins with linearizing the Weibull curve
  2. Follows with finding "Median Ranks"
  3. Ends with finding the parameters of the regression line then solving for eta and beta
  - Pros: credible solutions for small data sets
  - Cons: requires additional assumptions to address censored data

Maximum Likelihood Estimate

Newton's Method

$$x_{n+1} = x_n - \frac{f'(x_n)}{f''(x_n)}$$

Density Function

$$f(x) = \frac{\beta}{\eta}\left(\frac{x}{\eta}\right)^{\beta-1} e^{-(x/\eta)^{\beta}}$$

Likelihood Function

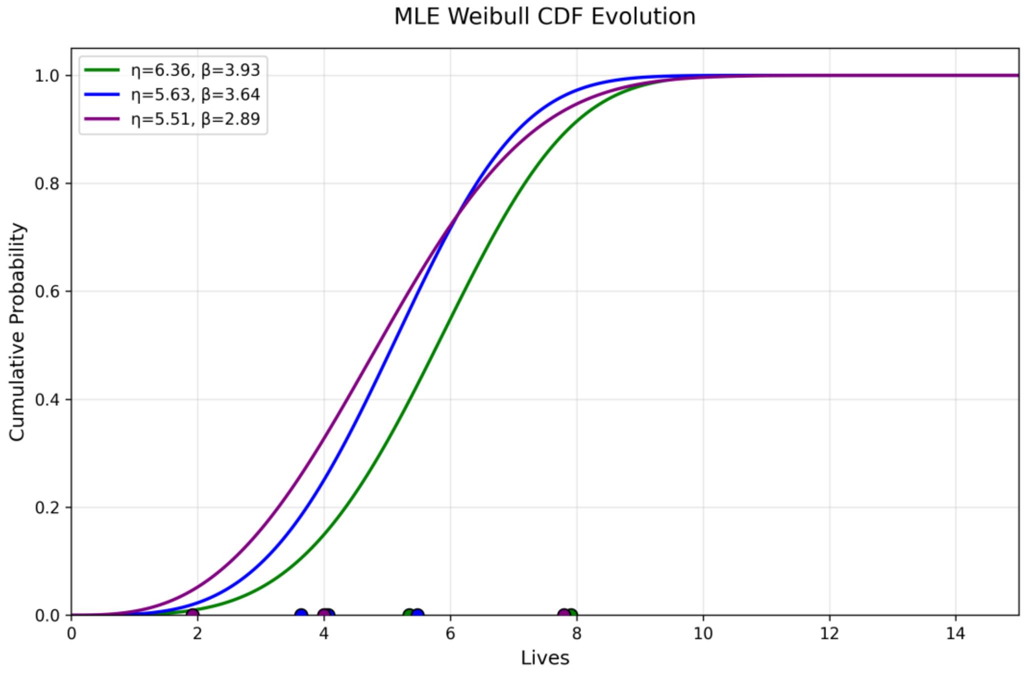
$$L(\eta,\beta) = \prod_{1}^{n} f(x_i)$$

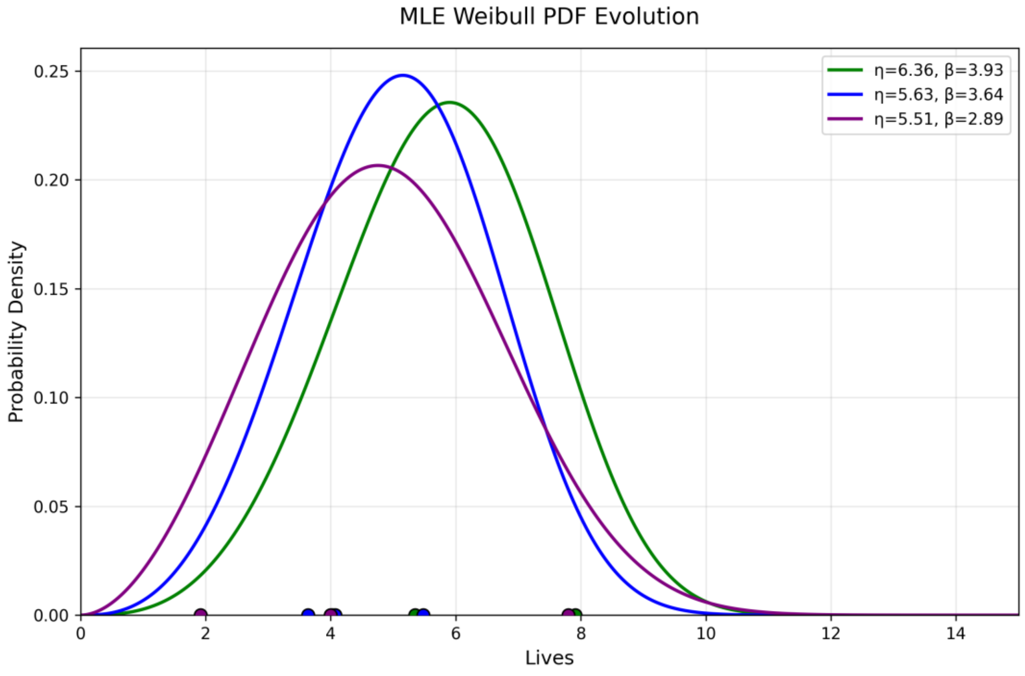
Log-Likelihood Function

$$LL(\eta,\beta) = \sum_{1}^{n} ln\big(f(x_i)\big)$$

Eta as a function of beta and data

$$\hat{\eta}(\hat{\beta}) = \frac{\frac{1}{n}\sum_1^n x_i}{\Gamma\left(1+\frac{1}{\hat{\beta}}\right)}$$





	x
x_1	4.03
x_2	7.91
x_3	5.35
x_4	3.64
x_5	5.48
x_6	4.07
x_7	4
x_8	7.8
x_9	1.92

Median Rank Regression

$$i - th\ Median\ Rank = F_{beta}(0.5, i, n+1-i)$$

Cumulative Probability

$$F(x) = 1 - e^{-(x/\eta)^{\beta}}$$

Linearized Cumulative Probability

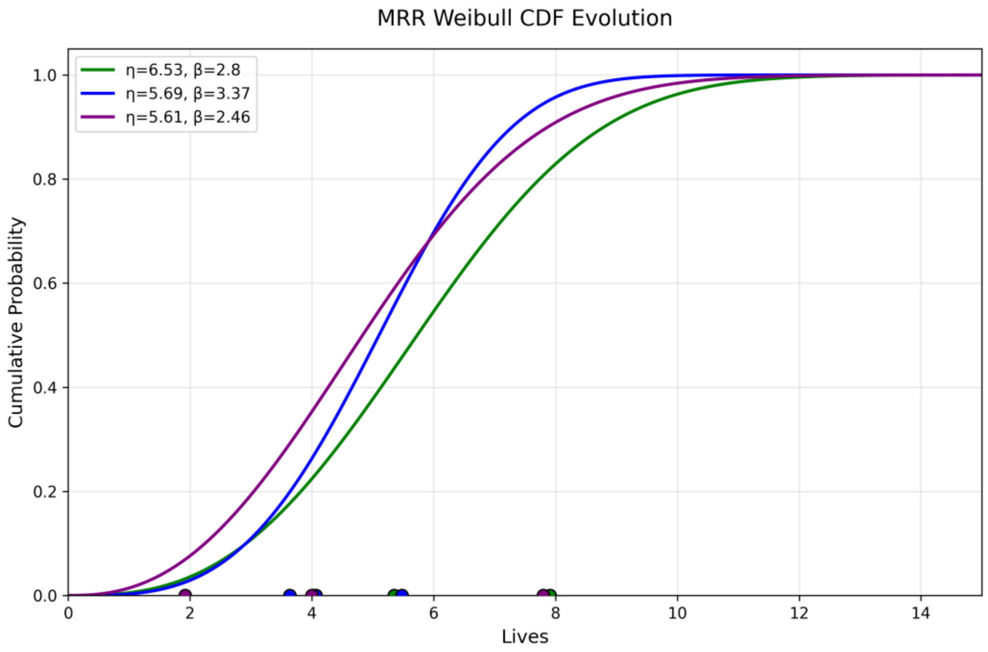
$$ln\left(-ln\left(1 - F(x)\right)\right) = \beta \cdot ln(x) - \beta \cdot ln(\eta)$$

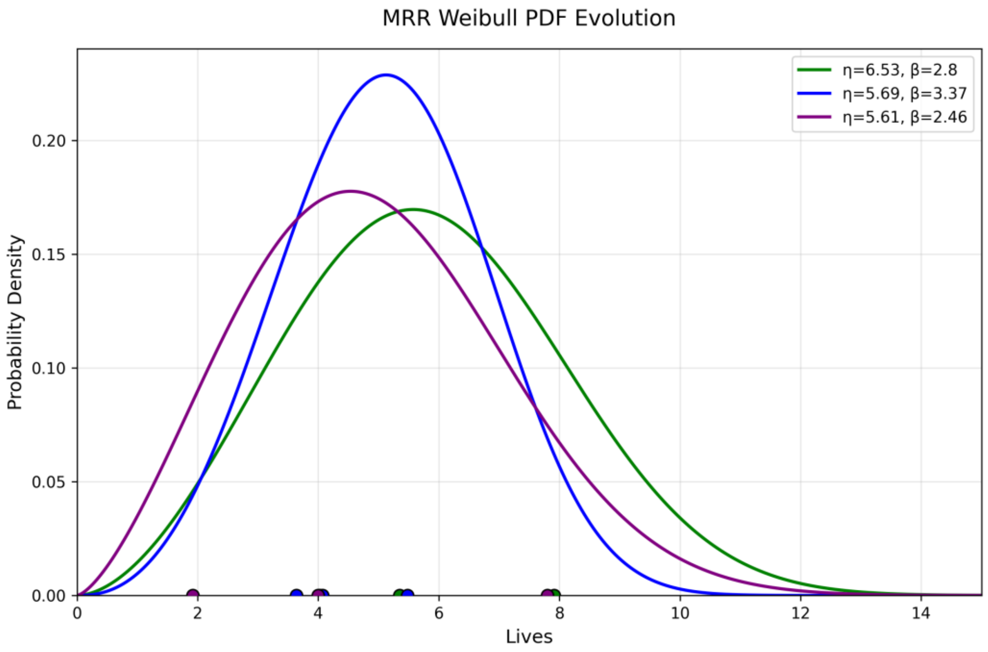
$$Y = m \cdot X + c$$

Regression Coefficients

$$m = \frac{\rho_{xy} \cdot \sigma_y}{\sigma_x}$$

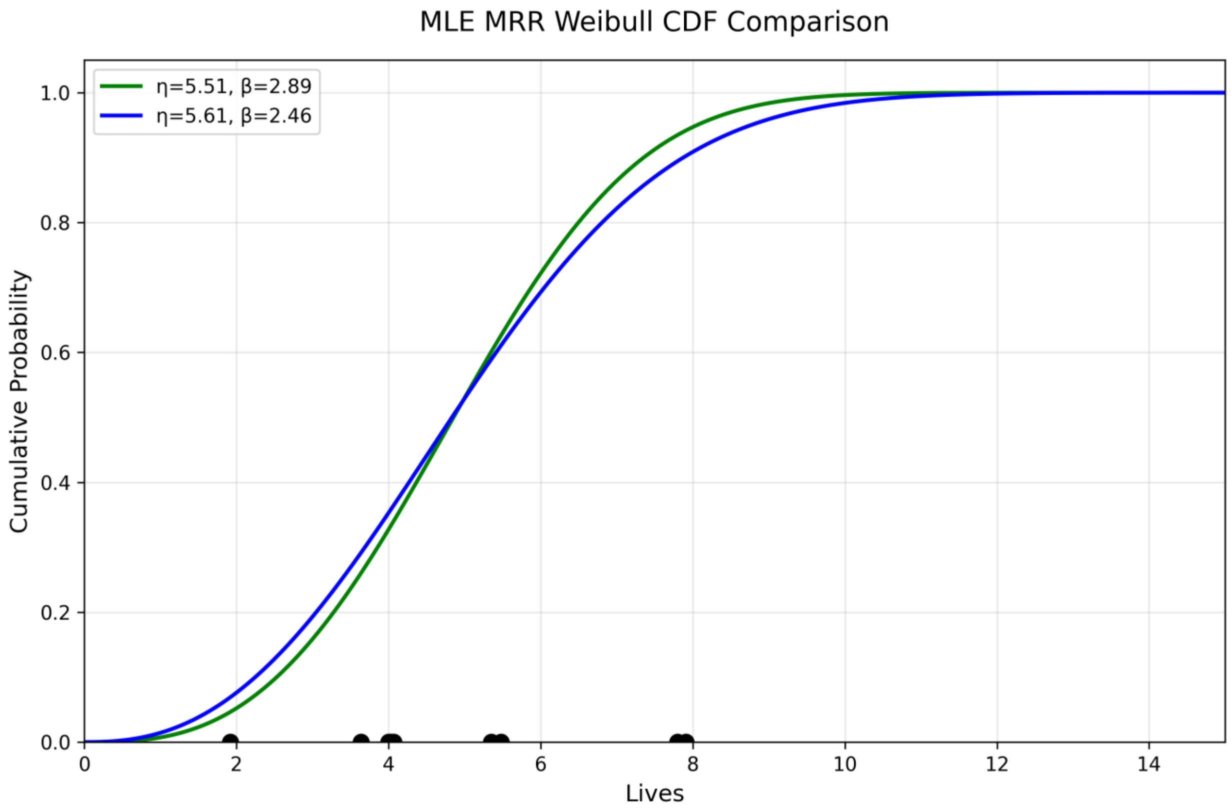
$$c = \mu_y - m \cdot \mu_x$$

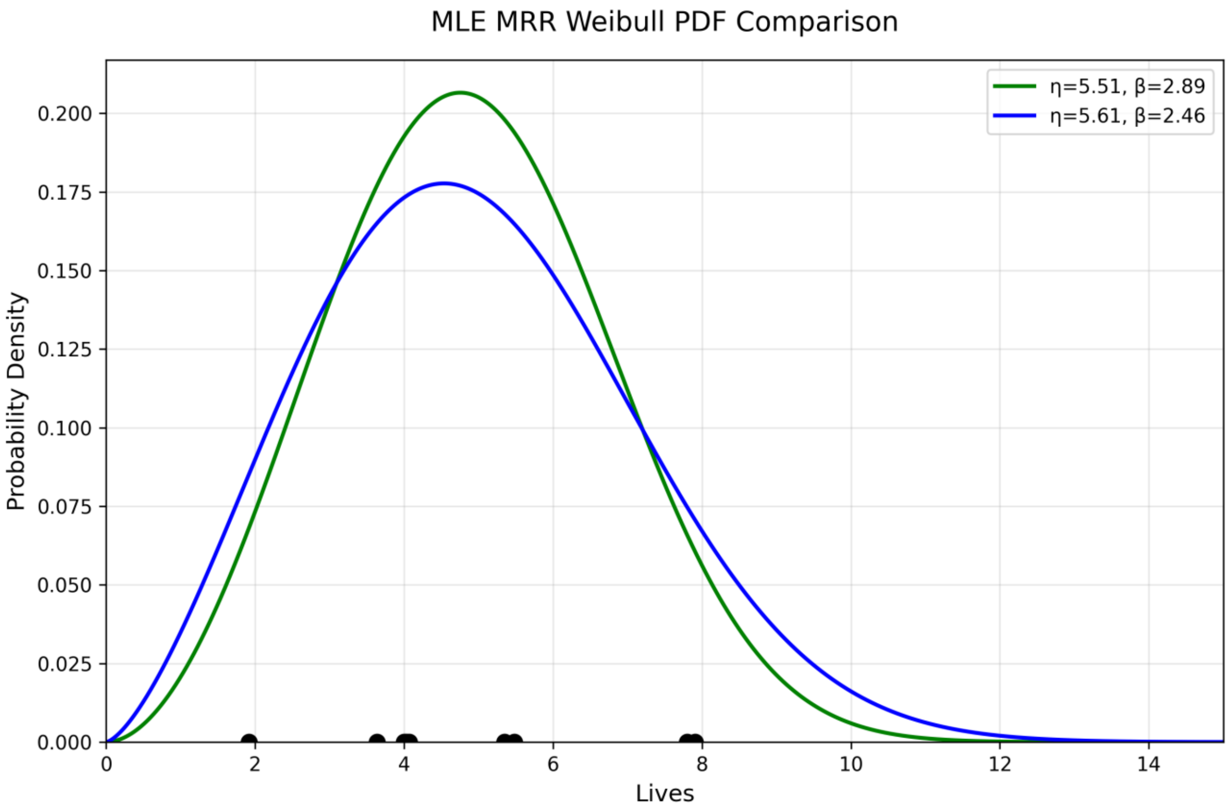




	x	Rank
x_1	4.03	4
x_2	7.91	9
x_3	5.35	6
x_4	3.64	2
x_5	5.48	7
x_6	4.07	5
x_7	4	3
x_8	7.8	8
x_9	1.92	1

MLE vs MRR


MLE MRR Weibull CDF Comparison
η=5.51, β=2.89
η=5.61, β=2.46
Cumulative Probability
Lives
0.0
0.2
0.4
0.6
0.8
1.0
0
2
4
6
8
10
12
14


MLE MRR Weibull PDF Comparison
η=5.51, β=2.89
η=5.61, β=2.46
Probability Density
Lives
0.000
0.025
0.050
0.075
0.100
0.125
0.150
0.175
0.200
0
2
4
6
8
10
12
14

Back up

Median Rank Table

	$i - th\ Median\ Rank = F_{beta}(0.5, i, n+1-i)$							
i	n = 2	n = 3	n = 4	n = 5	n = 6	n = 7	n = 8	n = 9
1	0.293	0.206	0.159	0.129	0.109	0.094	0.083	0.074
2	0.707	0.500	0.386	0.314	0.264	0.228	0.201	0.180
3		0.794	0.614	0.500	0.421	0.364	0.321	0.286
4			0.841	0.686	0.579	0.500	0.440	0.393
5				0.871	0.736	0.636	0.560	0.500
6					0.891	0.772	0.679	0.607
7						0.906	0.799	0.714
8							0.917	0.820
9								0.926